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Note

Acceleration of the Convergence in Viscous Flow Computations

In their paper [1], Gupta and Manohar thoroughly studied the influence of approximating the boundary conditions upon the accuracy of finite difference methods approaching the two-dimensional Navier–Stokes equations. These equations are written in stream function–vorticity formulation and the resulting system consists of two coupled, non-linear elliptic partial derivative equations (PDE).

The usual iterative procedure is employed: it requires a damping parameter for the calculation of the boundary values of the vorticity. In fact, this algorithm is of the relaxation type for both finite difference equations approaching the PDE, with two relaxation factors ω_1, ω_2, one of which is equal to unity in [1]. Previous work on the biharmonic problem has shown that the best rate of convergence is obtained when $\omega_1 = \omega_2$ [2, 5]. The numerical experiments described in this note allow this assertion to be verified for the finite difference approximations of Eqs. (1), (2).

The differential problem is written for a square cavity:

$$\Delta\psi = -\Omega \tag{1}$$

$$L\Omega = \Delta\Omega + R\left(\frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Omega}{\partial x}\right) = 0, \tag{2}$$

$$\psi = 0, \frac{\partial\psi}{\partial x} = 0, \qquad \text{when} \quad x = 0 \text{ or } 1,$$

$$\psi = 0, \frac{\partial\psi}{\partial y} = 0, \qquad \text{when} \quad y = 0, \tag{3}$$

$$\psi = 0, \frac{\partial\psi}{\partial y} = -1, \qquad \text{when} \quad y = 1.$$

The algorithm used is the following:

$$\Delta_h \bar{\psi}^{(m+1)} = -\Omega^{(m)} \tag{4}$$

$$\psi^{(m+1)} = \omega_1 \bar{\psi}^{(m+1)} + (1-\omega_1)\psi^{(m)} \qquad \text{on } (D), \tag{5}$$

$$\bar{\Omega}^{(m+1)} = f(\Omega^{(m)}, \psi^{(m)}) \tag{6}$$

$$\Omega^{(m+1)} = \omega_2 \bar{\Omega}^{(m+1)} + (1-\omega_2)\Omega^{(m)} \qquad \text{on } (\Gamma), \tag{7}$$

$$L_h \Omega^{(m+1)} = 0 \qquad \text{on } (D). \tag{8}$$

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Δ_h, L_h are the finite-difference operators associated with Δ and L. $f(\Omega^{(m)}, \psi^{(m)})$ is one of the boundary approximations detailed in [1].

It should be noted that the boundary conditions are explicitly calculated and damped by ω_2, in the usual way. $\Omega^{(m+1)}$ is then damped in the whole domain when Eq. (8) is solved. But (6) and (7) may be included in (8) and (6), (7), (8) replaced by (9):

$$\begin{aligned} L_h \bar{\Omega}^{(m+1)} &= 0 \\ \Omega^{(m+1)} &= \omega_2 \bar{\Omega}^{(m+1)} + (1-\omega_2)\,\Omega^{(m)} \end{aligned} \qquad \text{on } (\bar{D}). \tag{9}$$

Thus, the boundary conditions are implicitly taken into account.

The optimum rate of convergence of the above algorithm is obtained when [2],

$$\omega_1 = \omega_2 = \frac{2}{1+(1+\rho)^{1/2}}, \tag{10}$$

ρ being the growth factor of the outer iterative scheme (index (m) in (4)–(8)); ρ is obtained with a small number of iterations when $\omega_1 = \omega_2 = 1$. The values of ρ given in [1] are quite correct, so we obtain ω_1, ω_2 in Eq. (10) with these values.

The algorithm introduced by Smith [4] and utilized in [1] has a rate of convergence weaker than (4)–(8) when ω_1, ω_2 are given by (10). When the Reynolds number is large, the number of outer iterations strongly increases, each of them corresponding to an inner resolution for $\psi^{(m+1)}$ and $\Omega^{(m+1)}$ (by direct or iterative methods). To limit the increase in computing time, Gupta and Manohar suggest choosing larger values of p and q in the calculation of Ω on (Γ), i.e., to obtain Ω on the boundary using values at points not immediately next to the boundary. However, this choice leads to less accurate numerical results in the neighbourhood of the cavity corners and even for central points—see $\psi_{\max}$, Table III in [1].

For the same computing cost, the above algorithm (4)–(8) allows smaller values of p and q to be chosen thus improving the accuracy.

In Table I, we simply report some significant values of the number of outer iterations for the best-known boundary formulae:

Thom's formula ($p=1$, $q=0$), Jensen's formula ($p=2$, $q=1$), the formula with ($p=2$, $q=0$), Woods' formula.

This table should be compared with Table II in [1]. Note that in Eq. (8) we use central differencing for $R \leqslant 100$ and upstream differencing when $R = 500$.

The mesh size used in the numerical experiments is $h = 0.05$.

We would like to make some remarks concerning the calculation of $\psi^{(m+1)}$ and $\Omega^{(m+1)}$. For biharmonic problems direct methods seem to be preferable, but in our problem, the matrix corresponding to L_h changes at every iteration, therefore it is efficient to choose an iterative procedure: the point successive overrelaxation method for instance; but the choice of the relaxation factor is then difficult for (8), when R is large [3]: a strong under-relaxation is often necessary and the convergence is very slow.

TABLE I

Relaxation Factors and Number of Iterations N

Boundary formula	ω_1, ω_2	Reynolds number R				
		10	50	100	500	
$p=1, q=0$	0.46	22	32	43	71	N
$p=2, q=0$	0.58	15	20	26	43	N
$p=2, q=1$	0.39	29	40	56	88	N
Woods	0.45	21	34	47	76	N

However, it is not necessary to reach convergence of the iterative procedure: with a few iterations, 10 for instance, we obtain $\tilde{\psi}^{(m+1)}$, $\tilde{\Omega}^{(m+1)}$ which approximate to $\psi^{(m+1)}$, $\Omega^{(m+1)}$. This does not modify the convergence of the algorithm and the number of outer iterations. So it is probably possible to suggest a global iterative procedure which leaves out the outer iterations and decreases computing time.

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